Off-diagonal correlations of lattice impenetrable bosons in one dimension

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We consider off-diagonal correlation functions of impenetrable bosons on a lattice. By using the Jordan-Wigner transformation the one-body density matrix is represented as (Toeplitz) determinant of a matrix of fermionic Green functions. Using the replica method we calculate exactly the full long-range asymptotic behaviour of the one-body density matrix. We discuss how subleading oscillating terms, originating from short-range correlations give rise to interesting features in the momentum distribution.

I. INTRODUCTION

Correlation properties of one-dimensional bosons with short-range interactions is a long standing problem of statistical physics of integrable systems [1]. Despite the exact integrability, the correlation functions are hard to calculate due to a complicated form of the Bethe Ansatz wavefunctions and corresponding expressions for the matrix elements. Recently there has been a renewal of interest in these systems due to their relevance for current experiments with cold atoms. Experimental progress made it possible to create strongly correlated bosonic states in the gas phase [2, 3] and in an optical lattice [4, 5, 6], and approach the Girardeau-Tonks regime of impenetrable bosons. The physics of bosons in this regime is similar in many respects to that of free fermions [7], since the strong repulsive interactions effectively play a role of the Pauli principle. There is, however, an important difference due to bosonic quantum statistics of particles. The statistics reveals itself in off-diagonal correlations of bosons, such as the one-body density matrix or momentum distribution. The calculation of these observables in the Girardeau-Tonks limit from the first priciples represents an important question of statistical physics.

Impenetrable bosons in one dimension is an exactly soluble model both in the continuum case and on a lattice. In the latter case it is equivalent to the quantum XX spin chain: the two states of spin 1/2 on a given site correspond to the presence/absence of a boson. The correspondence with fermions was put forward in the early work by Lieb, Schultz and Mattis [8], where this system was mapped onto a system of free fermions on a lattice by the Jordan-Wigner transformation. Using this transformation it was possible to represent transverse spin correlation functions (corresponding to the off-diagonal correlations of bosonic operators) in the form of determinant of a matrix with a size given by the correlation distance. For a long time only an approximate behaviour of these objects was available. Schultz [9] has shown that at zero temperature correlation functions for large distances undergo a power law-decay. The explicit form of this power-law decay has been found by Efetov and Larkin [10] and has been recovered later by Vaidya and Tracy [11] using a method of mapping the quantum one-dimensional XX chain onto the two-dimensional classical Ising model. This technique has been succesfully employed in the works of McCoy, Barouch and Abraham [12] for calculating finite-temperature correlations. Recent numerical studies [13] of impenetrable bosons on a lattice in the presence of a confining potential have shown a high degree of universality of the power law decay found in the uniform case.

In this paper we reconsider the problem of calculating off-diagonal correlation functions of impenetrable bosons on a lattice. We start with the Jordan-Wigner transformation and represent the one-body density matrix as a determinant of a matrix given by fermionic Green functions. Next, we rewrite the determinant in terms of an average over the ensemble of unitary random matrices and calculate its asymptotic behaviour by employing the replica method. Since its first use to derive the pair distribution function for random matrices [14], the replica method has been successfully applied to strongly correlated systems such as Calogero-Sutherland models [15, 16] and impenetrable bosons in the continuum [17]. The essence of the replica method in this context is exactly the same as in the theory of disordered systems [18]. It consists of modifying the quantities being averaged so that they depend on a parameter n which is assumed to be integer.Once the average is performed, the result is recovered by a suitable analytical continuation to non-integer values of n. The main advantage of this method is that it leads to the asymptotic non-perturbative result

for the bosonic one-body density matrix including the leading power-law term plus oscillating contributions reflecting the short-range physics of the problem and lattice effects. This result allows one to determine the main features of the momentum distribution of impenetrable bosons on a lattice.

The paper is organised as follows: in Section II we use the Jordan-Wigner transformation to obtain the determinantal representation for the one-body density matrix of impenetrable bosons. We discuss how the results of Vaidya and Tracy are obtained by considering the leading asymptotic behaviour of the determinant. In Sections III and IV we present the replica method and derive our main result, the full asymptotic expansion of the one-body density matrix. Section V contains conclusions and prospects. The mathematical details are given in Appendices.

II. ONE-BODY DENSITY MATRIX AS A TOEPLITZ DETERMINANT

Let a_m, $a_m^\dagger$ be creation and annihilation operators of bosons on an infinite lattice with sites labeled by the index m. In the limit of strong repulsive on-site interactions the Hilbert space is projected onto a subspace of states with at most one particle in each site. Using the Jordan-Wigner transformation

$$a_m = \prod_{0<l<m} (-1)^{c_l^\dagger c_l} \, c_m; \qquad a_m^\dagger = c_m^\dagger \prod_{0<l<m} (-1)^{c_l^\dagger c_l} \tag{1}$$

the bosonic operators are related to fermionic creation and annihilation operators, $c_m^\dagger$, c_m obeying standard anti-commutation relations

$$\{c_m, c_l^\dagger\} = \delta_{m,l}; \qquad \{c_m, c_l\} = \{c_m^\dagger, c_l^\dagger\} = 0. \tag{2}$$

The one-body density matrix is defined by the following ground-state expectation value:

$$\begin{aligned}
g_1(R) &= \frac{1}{2}\langle (a_{m+R} + a_{m+R}^\dagger)(a_m + a_m^\dagger)\rangle \\
&= \frac{1}{2}\langle (c_{m+R} + c_{m+R}^\dagger) \prod_{m<l<m+R} (-1)^{c_l^\dagger c_l} \, (c_m - c_m^\dagger)\rangle,
\end{aligned} \tag{3}$$

where we have used the relation

$$(-1)^{c_l^\dagger c_l} = (c_l - c_l^\dagger)(c_l + c_l^\dagger) \tag{4}$$

and the fermionic commutation relations (2).

For $R = 0$ the one-body density matrix is related to the mean number of particles per site (filling factor) ν as

$$g_1(0) = \langle c_m^\dagger c_m \rangle - \frac{1}{2} = \nu - \frac{1}{2}, \tag{5}$$

while for $R > 1$ we use the relation (4) for every factor in the product in Eq. (3) and represent the one-body density matrix as an expectation value of $2R$ operators:

$$g_1(R) = \frac{1}{2}\langle A_{m+R} B_{m+R-1} A_{m+R-1} \ldots B_{m+1} A_{m+1} B_m\rangle, \tag{6}$$

where we have defined $A_l = c_l + c_l^\dagger$, $B_l = c_l - c_l^\dagger$. We now use the Wick theorem and take into account that free fermionic expectation values are given by

$$\begin{aligned}
\langle A_{m+l} B_m \rangle &\equiv G_l \\
\langle A_{m+l} A_m \rangle &= \langle B_{m+l} B_m \rangle = 0,
\end{aligned}$$

where the Green function of free fermions is

$$\begin{aligned}
G_l &= \langle (c_{m+l} + c_{m+l}^\dagger)(c_m - c_m^\dagger)\rangle = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iql} (n_q + n_{-q} - 1) \\
&= \frac{2}{\pi} \frac{\sin \pi \nu l}{l} - \delta_{l,0}.
\end{aligned} \tag{7}$$

Here $n_q = n_{-q}$ is the ground state momentum distribution of free fermions in a lattice with filling factor ν, so that

$$n_q + n_{-q} - 1 = \begin{cases} 1, & |q| < \pi\nu \\ -1, & |q| > \pi\nu \end{cases}. \tag{8}$$

Then the one-body density matrix (6) is represented as a determinant of $R \times R$ matrix

$$g_1(R) = \frac{1}{2} \begin{vmatrix} G_1 & G_2 & G_3 & \dots & G_R \\ G_0 & G_1 & G_2 & \dots & G_{R-1} \\ G_{-1} & G_0 & G_1 & \dots & G_{R-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{2-R} & G_{3-R} & G_{4-R} & \dots & G_1 \end{vmatrix}. \tag{9}$$

The determinant (9) has identical elements along each of its diagonals and thus belongs to the class of Toeplitz determinants:

$$g_1(R) = \det[g_{j-k}]_{j,k=1,\dots,R}; \qquad g_l = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(q) e^{iql} dq, \tag{10}$$

where $g(q)$ is called generating function (symbol). It is easily shown from (7) that in our case the generating function is

$$g(q) = e^{iq}(n_q + n_{-q} - 1) = \left(\frac{1 - e^{i\pi\nu} e^{-iq}}{1 - e^{-i\pi\nu} e^{iq}} \right)^{\frac{1}{2}} \left(\frac{1 - e^{-i\pi\nu} e^{-iq}}{1 - e^{i\pi\nu} e^{iq}} \right)^{\frac{1}{2}}. \tag{11}$$

It has two jump discontinuities at Fermi points $q = \pm\pi\nu$. The large size asymptotic behavior of Toeplitz determinants with such singular generating functions was a subject of extensive studies in mathematics (for overview and references see [19]). It has been shown for a certain class of singular generating functions that the large size asymptotics of Toeplitz determinants is correctly described by the Fisher-Hartwig conjecture [19]. In the case of generating function with two jumps as in Eq. (11) the analysis of the asymptotic behaviour of the Toeplitz determinant (9) is subtle and the Fisher-Hartwig asymptotic formula remains a conjecture. Being applied, it yields the following long-distance behaviour of the one-body density matrix:

$$g_1(R) = \frac{\rho_\infty}{\pi} \left| \frac{\sin \pi\nu}{R} \right|^{\frac{1}{2}}, \qquad R \to \infty, \tag{12}$$

where $\rho_\infty = \pi e^{1/2} 2^{-1/3} A^{-6}$, and $A = \exp(1/12 - \zeta'(-1)) \simeq 1.2824271$ is Glaisher's constant related to the Rieman zeta function. Numerical computation of the determinant (9) for $R \sim 100$ and various values of the filling factor shows that Eq. (12) provides an excellent estimate for the dominant smooth behaviour of the one-body density matrix at large distances. Substracting this dominant contribution from the exact numerical expression reveals interesting oscillating corrections. These corrections are sensitive to the specific value of the filling factor and reflect an interplay between short-distance interparticle correlations and lattice effects. The Fisher-Hartwig conjecture is unable to capture this behaviour. In the next Section we present the calculations of the asymptotic long-distance behaviour of the one-body density matrix, based on the replica method which has been recently applied to study correlation properties of integrable models. We shall see that this method is able to provide us with the dominant behaviour (12) as well as with the oscillating terms.

III. REPLICA CALCULATIONS OF THE ONE-BODY DENSITY MATRIX

Below we describe the replica method for finding the large R asymptotics of the determinant (9). By standard manipulations the determinant in (9) is transformed to the following R-dimensional integral:

$$g_1(R) = \frac{1}{2R!} \int_{-\pi}^{\pi} \frac{d^R q}{(2\pi)^R} |\Delta_R(e^{iq})|^2 \prod_{j=1}^{R} g(q_j) \equiv \frac{1}{2} \left\langle \prod_{j=1}^{R} g(q_j) \right\rangle_R, \tag{13}$$

where

$$\Delta_R(e^{iq}) = \Delta(e^{iq_1}, e^{iq_2}, \dots, e^{iq_R}) = \det[e^{i(k-1)q_l}]_{k,l=1,\dots,R} = \prod_{1 \le k < l \le R} (e^{iq_k} - e^{iq_l}) \tag{14}$$

is the Vandermonde determinant. The notation of average in (13) is justified by the fact that the measure of integration coincides with the distribution of eigenvalues of random unitary matrices drawn from Dyson's Circular Unitary Ensemble [20]. Denoting $\exp(iq_l) = z_l$ and $\exp(i\pi\nu) = v$ and using the representation (11) we arrive at an equation

$$
\begin{aligned}
g_1(R) &= \frac{1}{2} \left\langle \prod_{l=1}^{R} \left(\frac{1 - v\bar{z}_l}{1 - \bar{v}z_l} \right)^{\frac{1}{2}} \left(\frac{1 - \bar{v}\bar{z}_l}{1 - vz_l} \right)^{\frac{1}{2}} \right\rangle_R \\
&= \frac{1}{2} \left\langle \prod_{l=1}^{R} \frac{|1 - v\bar{z}_l|}{1 - \bar{v}z_l} \frac{|1 - \bar{v}\bar{z}_l|}{1 - vz_l} \right\rangle_R
\end{aligned}
\tag{15}
$$

The replica evaluation of the average (13) begins with modifying (replicating) the function being averaged in (15) by taking the $2n$-th power of absolute values. Then the original expression is recovered by taking $n \to 1/2$. It is however assumed throughout the calculations that n is an *integer* (and $2n$ is even) so that many simplifications occur. Eventually, the limit $n \to 1/2$ will be taken with care and we shall describe it in detail.

For an integer n, a straightforward algebra shows that we are dealing with an average of a rational function:

$$
\begin{aligned}
Z_n(R) &= \frac{1}{2} \left\langle \prod_{l=1}^{R} \frac{|1 - v\bar{z}_l|^{2n}}{1 - \bar{v}z_l} \frac{|1 - \bar{v}\bar{z}_l|^{2n}}{1 - vz_l} \right\rangle_R \\
&= \frac{v^{2nR}}{2} \left\langle \prod_{l=1}^{R} \frac{(\bar{v} - \bar{z}_l)^{2n}}{1 - \bar{v}z_l} \frac{(\bar{v} - z_l)^{2n}}{1 - vz_l} \right\rangle_R .
\end{aligned}
\tag{16}
$$

The crucial fact is the existance of a a duality transformation relating the R - dimensional integral (16) to a $2n-2 = m$ - dimensional integral:

$$
Z_n(R) = \frac{(-1)^{m+1}}{2S_m} \int_0^1 d^m x \Delta_m^2(x) \prod_{c=1}^{m} x_c(1 - x_c) \left[(1 - x_c)v + \bar{v}x_c \right]^R ,
\tag{17}
$$

where

$$
S_m = m! \prod_{c=1}^{m} \frac{\Gamma^2(1 + c)\Gamma(R + m + 1 - c)}{\Gamma(R + m + 2 + c)}
\tag{18}
$$

The proof of the duality transformation is presented in Appendix A.

Note that in the dual representation (17) large distance R appears only as a parameter, which makes the dual representation an excellent starting point for the asymptotic expansion of $g_1(R)$. The main contribution to the integral (17) comes from the end-points of integration. Let p be a number of variables in the vicinity of $x_- = 0$ and $p' = m - p$ be a number of variables close to $x_+ = 1$. We shift the integration variables and approximate the integrated high-degree polynomial as

$$
\begin{aligned}
x_c &= x_- + \frac{\xi_a}{R(1 - v^{-2})}, &\quad \left[(1 - x_c)v + \bar{v}x_c \right]^R &\simeq v^R e^{-\xi_c}, &\quad c &= 1, \ldots, p \\
x_{p+d} &= x_+ - \frac{\xi'_d}{R(1 - v^2)}, &\quad \left[(1 - x_{p+d})v + \bar{v}x_{p+d} \right]^R &\simeq v^{-R} e^{-\xi'_d}, &\quad d &= 1, \ldots, p'
\end{aligned}
\tag{19}
$$

The integration measure in (17) factorizes as

$$
\begin{aligned}
d^m x \, \Delta_m^2(x) \prod_{c=1}^{m} x_c(1 - x_c) &\simeq \left[\frac{1}{R(1 - v^{-2})} \right]^{p(p+1)} d^p \xi_c \, \Delta_p^2(\xi_c) \prod_{c=1}^{p} \xi_a \\
&\times \left[\frac{1}{R(1 - v^2)} \right]^{p'(p'+1)} d^{p'} \xi'_d \, \Delta_{p'}^2(\xi'_d) \prod_{d=1}^{p'} \xi'_d.
\end{aligned}
\tag{20}
$$

Then, summing over all possibilities to distribute m variables among saddle points and performing the remaining integration over ξ_c, ξ'_d we obtain

$$
Z_n(R) = \frac{(-1)^{m+1}}{S_m} \sum_{\substack{p,p' \\ p+p'=m}} \frac{m!}{p!p'!} I_p I_{p'} \left[\frac{1}{R(1 - v^{-2})} \right]^{p(p+1)} \left[\frac{1}{R(1 - v^2)} \right]^{p'(p'+1)} v^{R(p-p')},
\tag{21}
$$

where the integrals are performed using the Selberg integration formula

$$I_p = \int_0^\infty d^p \xi \, \Delta_p^2(\xi) \prod_{c=1}^p \xi_c e^{-\xi_c} = \prod_{c=1}^p \Gamma^2(1+c). \tag{22}$$

Combining them with the constant (18) we observe that each term in the sum (21) is proportional to the total combinatorial factor:

$$\frac{\Gamma^2(m+3)}{\Gamma(p+1)\Gamma(p'+1)} \left[F_{m+2}^{p+1} \right]^2 \prod_{c=1}^m \frac{\Gamma(R+m+2+c)}{\Gamma(R+m+1-c)}, \tag{23}$$

where

$$F_m^p = \frac{\prod_{d=1}^p \Gamma(d) \prod_{b=1}^{m-p} \Gamma(b)}{\prod_{c=1}^m \Gamma(c)} \tag{24}$$

is a familiar combination from previous studies of replica theories [14, 15].

Now we are at a position to take the limit $n \to 1/2$. It is important to notice that for even $m = 2n - 2$ there is always a central term $p = p' = m/2$ in the sum (21) which provides a smooth non-oscillatory contribution. It is characterized by a maximal degree of replica symmetry breaking, since it originates from the saddle point for which the number of components $x_d \sim x_-$ is equal to the number of components $x_b \sim x_+$. This term will be retained in the limit $n \to 1/2$ ($m \to -1$) and this provides an operational definition of the correct analytic continuation.

Making a change of summation index from p to $m/2 + k = n - 1 + k$ we factorize the combinatorial coefficient (24) as $F_{2n}^{n+k} = A_n D_k^{(n)}$, where

$$A_n = \prod_{c=1}^n \frac{\Gamma(c)}{\Gamma(2n+1-c)}, \tag{25}$$

$$D_k^{(n)} = \prod_{c=1}^k \frac{\Gamma(n+c)}{\Gamma(n+1-c)} = \frac{\Gamma(n+c)}{\Gamma(n+1-c)} D_{k-1}^{(n)}, \qquad k > 0; \qquad D_0^{(n)} = 1; \qquad D_k^{(n)} = D_{-k}^{(n)}. \tag{26}$$

This leads to an expression

$$Z_n(R) = \frac{C_n(R) A_n^2}{2} \left| \frac{1}{R(\bar{v}-v)} \right|^{2n(n-1)} \sum_{k=-\infty}^\infty \frac{\left[D_k^{(n)} \right]^2}{\Gamma(n+k)\Gamma(n-k)} \left| \frac{1}{R(\bar{v}-v)} \right|^{2k^2} \left(\frac{1-v^2}{1-\bar{v}^2} \right)^{(2n-1)k}. \tag{27}$$

Here we have defined

$$C_n(R) = (-1)^{2n-1} \Gamma^2(2n+1) \prod_{c=1}^{2n-2} \frac{\Gamma(R+2n+c)}{\Gamma(R+2n-1-c)} \tag{28}$$

and the summation over k in Eq. (27) was extended to all integers.

Now we take the limit $n \to 1/2$ term by term in the series (27). The only complication appears with the overall numerical factors C_n and A_n. The analytical continuation of these factors is explained in Appendix B where we show that $A_{\frac{1}{2}}^2 = \sqrt{2}\rho_\infty$ and $C_{\frac{1}{2}} = 1/R$. We also have $\Gamma(1/2+k)\Gamma(1/2-k) = \pi(-1)^k$. We therefore find that

$$g_1(R) = Z_{\frac{1}{2}}(R) = \frac{\rho_\infty}{\pi} \left| \frac{\sin \pi\nu}{R} \right|^{\frac{1}{2}} \left(1 + 2 \sum_{k=1}^\infty (-1)^k \left[D_k^{(\frac{1}{2})} \right]^2 \frac{\cos 2\pi k\nu R}{(2R\sin\pi\nu)^{2k^2}} \right). \tag{29}$$

The first several terms in this asymptotic expansion of the one-body density matrix read

$$g_1(R) = \frac{\rho_\infty}{\pi} \left| \frac{\sin \pi\nu}{R} \right|^{\frac{1}{2}} \left(1 - \frac{1}{8\sin^2\pi\nu} \frac{\cos 2\pi\nu R}{R^2} + \frac{9}{32768 \sin^8 \pi\nu} \frac{\cos 4\pi\nu R}{R^8} + \cdots \right), \tag{30}$$

where the leading term coincides with the result (12). As is seen from Eq. (30), for small k the coefficients in the series (29) rapidly decrease with increasing k. However, they will diverge for large k, which shows that we are dealing here with asymptotic, rather than convergent series.

IV. PERTURBATION THEORY

In Eq. (30), only the leading contribution of each saddle point was taken into account. To obtain the subleading terms one has to construct the perturbation theory around each saddle point. Using the shift of variables (19) the integrand in (17) is expanded in series in $1/R$. As a result, using the definition of the average

$$\left\langle f(\xi)g(\xi')\right\rangle_{p,p'} = \left\langle f(\xi)\right\rangle_p \left\langle g(\xi')\right\rangle_{p'} \qquad\qquad \left\langle f(\xi)\right\rangle_p = \frac{1}{I_p}\int_0^\infty d^p\xi\, f(\xi)\Delta_p^2(\xi)\prod_{c=1}^p \xi_c e^{-\xi_c} \tag{31}$$

each term in the series (21) gets multiplied by

$$\begin{aligned}
F_{p,p'}(R) = \Bigg\langle &\prod_{c=1}^p\prod_{d=1}^{p'}\left(1-\frac{\xi_c}{R(1-\bar{v}^2)}-\frac{\xi'_d}{R(1-v^2)}\right)\prod_{c=1}^p\left(1-\frac{\xi_c}{R(1-\bar{v}^2)}\right)\prod_{d=1}^{p'}\left(1-\frac{\xi'_d}{R(1-v^2)}\right)\\
&\times \prod_{c=1}^p e^{-\xi_c^2/2R-\xi_c^3/3R^2-\cdots}\prod_{d=1}^{p'}e^{-\xi'^2_d/2R-\xi'^3_d/3R^2-\cdots}\Bigg\rangle_{p,p'}
\end{aligned} \tag{32}$$

Up to terms of the order of $1/R^2$, it is sufficient to put $p = m/2 = -1/2$, since the leading contribution of other saddle points is of the order of $1/R^2$ or higher. The function $F_{m/2,m/2}(R)$ is then evaluated using the following averages (see chapter 17 in [20]) :

$$\left\langle \xi_1\right\rangle_p = p+1,$$

$$\left\langle \xi_1^2\right\rangle_p = (2p+1)(p+1), \qquad\qquad \left\langle \xi_1\xi_2\right\rangle_p = p(p+1)$$

$$\left\langle \xi_1^3\right\rangle_p = (p+1)(5p^2+5p+2), \qquad\qquad \left\langle \xi_1^2\xi_2\right\rangle_p = 2p^2(p+1)$$

$$\left\langle \xi_1^4\right\rangle_p = (2p+1)(p+1)(7p^2+7p+2) \qquad\qquad \left\langle \xi_1^2\xi_2^2\right\rangle_p = 2p^2(2p-1)(p+1). \tag{33}$$

The result then is

$$F_{-1/2,-1/2}(R) = 1 - \frac{\cot^2\pi\nu}{32R^2} \tag{34}$$

which leads to the following asymptotic expression for the one-body density matrix:

$$g_1(R) = \frac{\rho_\infty}{\pi}\left|\frac{\sin\pi\nu}{R}\right|^{\frac{1}{2}}\left(1-\frac{\cot^2\pi\nu}{32R^2}-\frac{1}{8\sin^2\pi\nu}\frac{\cos 2\pi\nu R}{R^2}+\dots\right) \tag{35}$$

Our result reproduces correctly the continuous limit. In this case it is useful to introduce a length scale, the lattice constant a, so that $x = Ra$ is the distance and $\pi\nu/a = k_F$ is the Fermi wavenumber proportional to the density. Going to the continuous limit corresponds to having the filling factor ν tending to zero while the product $k_F x = \pi\nu R$ is kept fixed. Dividing g_1 by the lattice constant to have a correct normalization, Eq. (30) becomes

$$\tilde{g}_1(x) = (1/a)g_1(x/a) = \frac{\rho_\infty}{|k_F x|^{\frac{1}{2}}}\left(1-\frac{1}{32(k_F x)^2}-\frac{1}{8}\frac{\cos 2k_F x}{(k_F x)^2}+\dots\right) \tag{36}$$

in agreement with the Vaidya and Tracy result [21]. The sign of the oscillating cosine term in Eq. (36) has been corrected as discussed in [17].

V. CONCLUSIONS AND PROSPECTS

The result (35) provides an exact analyical expression for the long distance behaviour of the one-body density matrix for impenetrable bosons on an infinite lattice. It is valid for all values of the filling factor ν, in particular it reproduces correctly the continuous limit $\nu \to 0$. The structure of Eq. (35) is in accordance with the hydrodynamic expansion

conjectured by Haldane [22]. Equations (25), and (26) (at $n = 1/2$) provide the exact values for the non-universal coeficients of the leading oscillating terms.

The oscillatory terms in (35) reflect the physics on short distance scales, of the order of the mean interparticle separation. They are analogous to the Friedel oscillations in the physics of fermions and provide yet another manifestation of fermionization. The interplay between these short-distance correlations and lattice effects is interesting, for instance, in the case of half filling ($\nu = 1/2$), where the particles tend to occupy every second site. This results in the oscillations of the one-body density matrix with the period of the lattice.

The momentum distribution cannot be reconstructed from the asymptotic expression (35) for the one-body density matrix, as it requires the knowledge of the short-distance behaviour. However, we can predict some of the important features. For sufficiently small momenta their distribution diverges as an inverse square root of the momentum. The finite size would result in the finite peak value of the momentum distribution proportional to the square root of the total number of particles. For momenta of the order of $2k_F = 2\pi\nu/a$ the derivative of the momentum distribution has a jump (the distribution itself has a cusp) resulting from the underlying Fermi surface of the fermionized bosons. Weaker singularities exist for higher multiples of $2k_F$. It is again interesting to see that for the case of half filling the jump in the derivative occurs just at the border of the Brillouin zone and therefore cannot be observed: the momentum distribution has no singularities.

These features can be probed in current experiments with cold atoms. The most promising method is expected to be a combination of Bragg spectroscopy with the time of flight technique. In this case a Bragg pulse is applied after switching off the trap, when the density of the cloud is already sufficiently small so that the number of scattered atoms (proportial to the momentum disitribution) can be detected with precision [23]. This technique can allow the observation of the singularities such as cusps in the momentum distribution originating from the oscillating subleading terms in the coordinate correlation functions.

There are several directions in which the present work can be extended. An immediate question concerns the finite value of the interactions or, in other words the case of the soft-core bosons. This case is similar to the case of the Heisenberg spin chain, for which there are recent results based on the Bethe Ansatz solution [24]. It is not clear at the moment how to obtain the explicit behaviour of the correlation functions away from the free fermionic point (equivalent to the hard-core bosons) where the Toeplitz determinant expression exists for the one-body density matrix. In the case of impenetrable bosons, our work can be extended to the domain of time-dependent correlation functions which contains information about elementary excitations and are related to the responce of the system to external perturbations.

APPENDIX A: DUALITY TRANSFORMATION

The average (16) is symmetric under the interchange of v and $\bar{v}$, so that using the fact that $z_l^{-1} = \bar{z}_l$ it is possible to bring Eq. (16) to the form

$$
\begin{aligned}
Z_n(R) &= \frac{(-1)^R}{2} \left\langle \prod_{l=1}^{R} \bar{z}_l (1 - \bar{v}\bar{z}_l)^{2n-1} (v - z_l)^{2n-1} \right\rangle \\
&= \lim_{\substack{v_1 \to \infty,\ v_2,\dots,v_{2n} \to v \\ \bar{u}_1,\dots,\bar{u}_{2n-1} \to v,\ \bar{u}_{2n} \to 0}} \frac{\prod_{c=1}^{2n} v_c^{-R}}{2} \left\langle \prod_{l=1}^{R} \prod_{c=1}^{2n} (\bar{u}_c - \bar{z}_l)(v_c - z_l) \right\rangle,
\end{aligned} \tag{A1}
$$

where we have "split" $2(2n-1)$ points in the product using variables $\bar{u}_c$ for $c = 1, \dots, 2n-1$ and v_c for $c = 2, \dots, 2n$. We have also defined two additional variables v_1 and $\bar{u}_{2n}$ in order to represent the factor z_l in the product on the same footing.

Recalling the definition (13) for the average and using an obvious relation

$$\prod_{l=1}^{R}\prod_{c=1}^{2n}(v_c - z_l) = \frac{\Delta_{R+2n}(v_1, \ldots, v_{2n}, z_1, \ldots, z_R)}{\Delta_{2n}(v_1, \ldots, v_{2n})\Delta_R(z_1, \ldots, z_R)} \tag{A2}$$

we rewrite Eq. (16) as a many-body correlation function

$$\begin{aligned}
Z_n(R) &= \frac{\prod_{c=1}^{2n} v_c^{-R}}{2\Delta_{2n}(\bar{u})\Delta_{2n}(v)} \frac{1}{R!} \int_{-\pi}^{\pi} \frac{d^R q}{(2\pi)^R} \Delta_{R+2n}(\bar{u}_1, \ldots, \bar{u}_{2n}, \bar{z}_1, \ldots, \bar{z}_R)\Delta_{R+2n}(v_1, \ldots, v_{2n}, z_1, \ldots, z_R) \\
&= \frac{\prod_{c=1}^{2n} v_c^{-R}}{2\Delta_{2n}(\bar{u})\Delta_{2n}(v)} \langle 0|\psi^\dagger(u_1)\ldots\psi^\dagger(u_{2n})\psi(v_{2n})\ldots\psi(v_1)|0\rangle \tag{A3}
\end{aligned}$$

of fermionic creation and annihilation operators $\psi^\dagger(\bar{s}_c)$, $\psi(\bar{t}_c)$ in the ground state of $R+2n$ fermions. Up to normalization, the ground state wavefunction is given by the Vandermonde determinants: $\langle z|0\rangle \propto \Delta_{R+2n}(z)$. The expectation value of the fermionic operators is given by the Wick theorem in the form of $2n \times 2n$ determinant:

$$\langle 0|\psi^\dagger(u_1)\ldots\psi^\dagger(u_{2n})\psi(v_{2n})\ldots\psi(v_1)|0\rangle = \begin{vmatrix}
G(\bar{u}_1 v_1) & G(\bar{u}_1 v_2) & G(\bar{u}_1 v_3) & \ldots & G(\bar{u}_1 v_{2n}) \\
G(\bar{u}_2 v_1) & G(\bar{u}_2 v_2) & G(\bar{u}_2 v_3) & \ldots & G(\bar{u}_2 v_{2n}) \\
G(\bar{u}_3 v_1) & G(\bar{u}_3 v_2) & G(\bar{u}_3 v_3) & \ldots & G(\bar{u}_3 v_{2n}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
G(\bar{u}_{2n} v_1) & G(\bar{u}_{2n} v_2) & G(\bar{u}_{2n} v_3) & \ldots & G(\bar{u}_{2n} v_{2n})
\end{vmatrix} \tag{A4}$$

with the following 2-fermionic Green function:

$$G(\bar{u}v) = \langle 0|\psi^\dagger(u)\psi(v)|0\rangle = \sum_{p=0}^{R+2n-1} \bar{u}^p v^p = \frac{1 - (\bar{u}v)^{R+2n}}{1 - \bar{u}v}. \tag{A5}$$

We now take the limits $\bar{u}_{2n} \to 0$, $v_1 \to \infty$ using $G(\bar{u}_{2n}v) = G(0) = 1$ and $G(\bar{u}v_1) \simeq (\bar{u}v_1)^{R+2n-1}$. The factor v_1^{R+2n-1} common to the elements in the first column cancels with the corresponding term in the prefactor of the second line of Eq. (A1). Taking out a factor $\bar{u}_c^{R+2n-1}$ from each row we arrive at the following representation:

$$Z_n(R) = -\frac{1}{2}\frac{\prod_{c=1}^{2n-1} \bar{u}_c^R v_{c+1}^{-R}}{\Delta_{2n-1}(u)\Delta_{2n-1}(v)} \begin{vmatrix}
1 & \widetilde{G}(u_1, v_2) & \widetilde{G}(u_1, v_3) & \ldots & \widetilde{G}(u_1, v_{2n}) \\
1 & \widetilde{G}(u_2, v_2) & \widetilde{G}(u_2, v_3) & \ldots & \widetilde{G}(u_2, v_{2n}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \widetilde{G}(u_{2n-1}, v_2) & \widetilde{G}(u_{2n-1}, v_3) & \ldots & \widetilde{G}(u_{2n-1}, v_{2n}) \\
0 & 1 & 1 & \ldots & 1
\end{vmatrix}, \tag{A6}$$

where $\widetilde{G}(u, v) = (u^{R+2n} - v^{R+2n})/(u - v)$ and we have used the relation $\Delta_m(\bar{u}_1, \ldots, \bar{u}_m) = -\prod_{a=1}^{m} \bar{u}_a^{m-1}\Delta_m(u_1, \ldots, u_m)$.

We take the singular limit $u_c \to u$ and $v_c \to v$ by a standard procedure, described, for example in [25]. Expanding in non-zero elements of the last row and first column leads to expressing Eq. (A6) as a determinant of $(2n-2)\times(2n-2)$ matrix of partial derivatives:

$$Z_n(R) = \frac{(-1)^{2n-1}}{2\prod_{c=1}^{2n-1}\Gamma^2(c)} \begin{vmatrix}
\partial_u\partial_v\widetilde{G} & \partial_u\partial_v^2\widetilde{G} & \ldots & \partial_u\partial_v^{2n-2}\widetilde{G} \\
\partial_u^2\partial_v\widetilde{G} & \partial_u^2\partial_v^2\widetilde{G} & \ldots & \partial_u^2\partial_v^{2n-2}\widetilde{G} \\
\vdots & \vdots & \ddots & \vdots \\
\partial_u^{2n-2}\partial_v\widetilde{G} & \partial_u^{2n-2}\partial_v^2\widetilde{G} & \ldots & \partial_u^{2n-2}\partial_v^{2n-2}\widetilde{G}
\end{vmatrix} \tag{A7}$$

Now we use the following integral representation in which we recognize the expression for the hypergeometric function:

$$\begin{aligned}
\partial_u\partial_v\widetilde{G}(u, v) &= \frac{\Gamma(R+2n+1)}{\Gamma(R+2n-2)} \int_0^1 dx\, x(1-x)[(1-x)u + vx]^{R+2n-3} \\
&= \frac{\Gamma(R+2n+1)}{\Gamma(R+2n-2)}\frac{\Gamma^2(2)}{\Gamma(4)} u^{R+2n-3}F(-R-2n+3, 2; 4; s), \tag{A8}
\end{aligned}$$

where $s = 1 - \bar{u}v$. Splitting again the variables $s \to s_c = 1 - \bar{u}_c v$, for $c = 1, \ldots, 2n - 2$ the determinant (A7) is represented once again as a ratio

$$
Z_n(R) = \frac{(-1)^{2n-1}}{2\Gamma^2(2n-1)\prod_{c=1}^{2n-2}\Gamma(c)} \left[\frac{\Gamma(R+2n+1)}{\Gamma(R+2n-2)}\frac{\Gamma^2(2)}{\Gamma(4)}\right]^{2n-2} \frac{\prod_{c=1}^{2n-2} u_c^R}{\Delta_{2n-2}(s)}
$$

$$
\times \begin{vmatrix} F_0(s_1) & F_1(s_1) & \ldots & F_{2n-3}(s_1) \\ F_0(s_2) & F_1(s_2) & \ldots & F_{2n-3}(s_2) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(s_{2n-2}) & F_1(s_{2n-2}) & \ldots & F_{2n-3}(s_{2n-2}) \end{vmatrix}, \tag{A9}
$$

where $F_k(s) = [(s-1)\partial_s]^k F(-R - 2n + 3, 2; 4; s)$. In this function k derivatives ∂_s can be commuted through the factors $s - 1$ to the right, producing additional terms which has at most $k - 1$ derivatives. These terms do not contribute to the determinant, since they already appear in the $k-1$-th column. Next, we use the known properties of hypergeometric functions to show that

$$
F_k(s) = (s-1)^k \partial_s^k F(-R - 2n + 3, 2, 4, s) = \frac{(-R - 2n + 3)_k (2)_k}{(4)_k}(s-1)^k F(-R - 2n + 3 + k, 2 + k; 4 + k; s)
$$

$$
= \frac{\Gamma(R+2n-2)}{\Gamma(R+2n-2-k)}\frac{\Gamma(2+k)}{\Gamma(2)}\frac{\Gamma(4)}{\Gamma(4+k)}(1-s)^{R+2n-1} F(R+2n+1, 2; 4+k; z).
$$

Substituting this expression into (A9) and taking the limit $s_c \to s = 1 - \bar{u}v$ we get

$$
Z_n(R) = \frac{(-1)^{2n-1}}{2} \prod_{c=1}^{2n-2} \frac{\Gamma(2)\Gamma(R+2n+1)}{\Gamma(1+c)\Gamma(3+c)\Gamma(R+2n-1-c)}(1-s)^{(n-1)(R+4n-2)}
$$

$$
\times \begin{vmatrix} \widetilde{F}(0;s) & \widetilde{F}(1;s) & \ldots & \widetilde{F}(2n-3;s) \\ \partial_s\widetilde{F}(0;s) & \partial_s\widetilde{F}(1;s) & \ldots & \partial_s\widetilde{F}(2n-3;s) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_s^{2n-3}\widetilde{F}(0;s) & \partial_s^{2n-3}\widetilde{F}(1;s) & \ldots & \partial_s^{2n-3}\widetilde{F}(2n-3;s) \end{vmatrix}, \tag{A10}
$$

where $\widetilde{F}(p;s) = F(R + 2n + 1, 2; 4 + p; s)$. For the derivatives of the function $\widetilde{F}_p(s)$ we have

$$
\partial_s^k \widetilde{F}(p;s) = \frac{(N+2n+1)_k(2)_k}{(4+p)_k}F(N+2n+1+k, 2+k; 4+p+k; s)
$$

$$
= \frac{(R+2n+1)_k(2)_k}{(4+p)_k}(1-s)^{-R-2n+1+p-k}F(-R-2n+3+p, 2+p; 4+p+k; s)
$$

$$
= \frac{\Gamma(R+2n+1+k)}{\Gamma(R+2n+1)}\frac{\Gamma(4+p)}{\Gamma(2)\Gamma(2+p)}(1-s)^{-R-2n+1+p-k}\int_0^1 dx\, x^{p+1}(1-x)^{k+1}(1-sx)^{R+2n-3-p}.
$$

In this expression we used the properties of hypergeometric functions [26] along with its integral representation. Substituting this result into (A10) yields

$$
Z_n(R) = \frac{(-1)^{2n-1}}{2} \prod_{c=1}^{2n-2} \frac{\Gamma(R+2n+c)}{\Gamma^2(1+c)\Gamma(R+2n-1-c)}(1-s)^{R(n-1)}
$$

$$
\times \int_0^1 dx_1 \ldots dx_{2n-2}\, (1-x_1)^0 (1-x_2)^1 \ldots (1-x_{2n-2})^{2n-3} \prod_{c=1}^{2n-2} x_c(1-x_c)
$$

$$
\times \begin{vmatrix} (1-sx_1)^{R+2n-3} & x_1(1-sx_1)^{R+2n-4} & \ldots & x_1^{2n-3}(1-sx_1)^R \\ (1-sx_2)^{R+2n-3} & x_2(1-sx_2)^{R+2n-4} & \ldots & x_2^{2n-3}(1-sx_2)^R \\ \vdots & \vdots & \ddots & \vdots \\ (1-sx_{2n-2})^{R+2n-3} & x_{2n-2}(1-sx_{2n-2})^{R+2n-4} & \ldots & x_{2n-2}^{2n-3}(1-sx_{2n-2})^R \end{vmatrix}. \tag{A11}
$$

The first factor in the integrand is anti-symmetrized to yield Vandermonde determinant $\Delta_{2n-2}(1-x) = -\Delta_{2n-2}(x)$. The integration can be performed column-wise. This enables one to extract another factor $-\Delta_{2n-2}(x)$. Finally, setting $m = 2n - 2$ and $1 - s = v^2$ and interchanging v and $\bar{v}$ we obtain Eq. (17).

APPENDIX B: ANALYTICAL CONTINUATION OF PRODUCTS A_n AND $C_n(R)$

The main idea behind evaluating the products $\prod_{c=1}^{n} f_c$ such as (25) and (28) for fractional or negative values of the product limit n is to turn the products into exponentials of sums of logarithms of each factor and then use the integral representation [26] for the logarithm of Euler's gamma function. It is possible to perform the summation under the integral and obtain an integral representation where n enters only as a parameter. The resulting integral representation can be analytically continued to a desired value of n. In the case of A_n the procedure is described in detail in Appendix B of Ref. [17]. We cite here the final result:

$$\rho_\infty = A_{1/2}^2/\sqrt{2} = \pi e^{1/2} 2^{-1/3} A^{-6}. \tag{B1}$$

For the function $C_n(R)$ defined by Eq. (28) we have

$$
\begin{aligned}
\ln \frac{C_n(R)}{(-1)^{2n-1}\Gamma^2(2n+1)} &= \sum_{c=1}^{2n-2} \left(\ln\Gamma(R+2n+c) - \ln\Gamma(R+2n-1-c) \right) \\
&= \int_0^\infty \frac{dt}{t} \left[e^{-Rt} \sum_{c=1}^{2n-2} \frac{e^{-(2n+c)t} - e^{-(2n-1-c)t}}{1-e^{-t}} + \sum_{c=1}^{2n-2} (2c+1)e^{-t} \right] \\
&= \int_0^\infty \frac{dt}{t} \left[e^{-Rt} \frac{e^{-(2n+1)t} - e^{-(4n-1)t} + e^{-(2n-1)t} - e^{-t}}{(1-e^{-t})^2} + 2n(2n-2)e^{-t} \right]
\end{aligned} \tag{B2}
$$

For $n = 1/2$ the first term in square brackets is exactly equal to e^{-Rt}, and we obtain the desired analytical continuation

$$\ln C_{\frac{1}{2}}(R) = \int_0^\infty \frac{dt}{t} \left(e^{-Rt} - e^{-t} \right) = \ln \frac{1}{R}. \tag{B3}$$

Thus $C_{\frac{1}{2}}(R) = 1/R$.

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